

Structure and Representation Theory for Double Group of Four-Dimensional Cubic Group

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Abstract

Hyper-cubic groups in any dimension are defined and their conjugate classifications and representation theories are derived. Double group and spinor representation are introduced. A detailed calculation is carried out on the structures of four-dimensional cubic group O_4 and its double group, as well as all inequivalent single-valued representations and spinor representations of O_4 . All representations are derived adopting Clifford theory of decomposition of induced representations. Based on these results, single-valued and spinor representations of the orientation-preserved subgroup of O_4 are calculated.

Keywords: hyper-cubic group, double group, spinor representation, Clifford theory

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I Introduction

It is well-known that electrons stay in spinor representations of the symmetry group of a given lattice in condensed matter physics; it is reasonable to assume that quarks, leptons, as well as baryons, should reside in *spinor representations* of the symmetry group of a four-dimensional lattice in lattice field theory (the concept of “spinor representation” will be clarified in the next section). Accordingly, to explore the structure and representations (spinor representations especially) of such groups has important significance in high energy physics.

In this paper, we concentrate on the case of hyper-cubic lattices, though they are not the maximum symmetric lattices in four dimension [1]. In history, the first representation-theoretical consideration of symmetry group of such lattices was given by A. Young [2]. Then mathematicians worked in this field due to the interest of wreath product [3][4] to which A. Kerber gave a thorough review in his book [5]. Physicists took part in after K. G. Wilson introduced lattice gauge theory [6]. M. Baake *et al* first gave an explicit description of characters of four-dimensional cubic group [7]; J. E. Mandula *et al* derived the same results using a different method [8]. As for spinor representations, Mandula *et al* resolved this problem for what we call *orientation-preserved* four-dimensional cubic-group in [9].

In this paper, the power of Clifford theory on decomposition of induced representations (Sec.II.1) is fully applied. A systematic and schematic description of conjugate classification and representation theory of generalized cubic group O_n , as well as the concept of orientation-preserved subgroup of them SO_n , is given in Sec.II.2. Double group is introduced in Sec.II.3 to clarify the terms “single-valued representation” and “two-valued representation (spinor representation)”. Then specifying these general results to four dimension, we give a detailed description of structure and conjugate classification of O_4 (Sec.III.1), its double \overline{O}_4 (Sec.III.2), and those of SO_4 , \overline{SO}_4 (Sec.V). We derive all inequivalent single-valued representations as well as spinor representations of O_4 , adopting Clifford theory (Sec.IV). Based on these results, we reproduced representation theory of \overline{SO}_4 in Sec.VI.

It should be pointed out that the “spinor” part for O_4 of our work is completely new and that although other results are well-known, our method to derive them is much more tidy and systematic than that used by other authors who gave the same results, thanks for the power of Clifford theory.

II Conceptual foundations

II.1 Clifford theory on decomposition of induced representations

Two results of Clifford theory, a powerful method for decomposing induced representations of a given group E with a normal subgroup N [10][11][12][13], will be applied in this paper. We will use $\mathcal{C}[E]$ for the group algebra of E in complex field and G for E/N below. The first result is

Theorem 1 (*Clifford*)[14][10] *Let M be a simple $\mathcal{C}[E]$ -module, and L a simple $\mathcal{C}[N]$ -submodule of M_N s.t. L is stable relative to E , i.e. L is isomorphic to all of its conjugates. Then*

$$M \cong L \otimes_{\mathcal{C}} I$$

for a left ideal I in $\text{End}_{\mathcal{C}[E]} L^E$. The E -action on $L \otimes_{\mathcal{C}} I$ is given by

$$x \mapsto U(x) \otimes V(x), x \in E$$

where $U : E \rightarrow GL(L)$ is a projective representation of E on L , and $V : E \rightarrow GL(I)$ is a projective representation of G , that is, $V(x)$ depends only on the coset xN of x in G , for each $x \in E$. The factor sets associated with U and V are inverse of each other.

The second result can be regarded as a special case of Theorem 1. Let $E = N \rtimes G$, $|E| < \infty$ and N be abelian, then adjoint action of G upon N makes N a G -module. This G -action can be extended naturally to a G -action upon $\mathcal{C}[N]$ by linearity. Define $\Pi(N) := \{\pi_\mu\} \subset \mathcal{C}[N]$,

$$\pi_\mu := \sum_{a \in N} \chi_\mu(a^{-1})a \tag{1}$$

where χ_μ are all irreducible representations of N . The G -action on $\Pi(N)$ is closed and thus $\Pi(N)$ is separated into orbits $\Pi(N) = \coprod_{o \in \mathcal{I}} \Pi_o$ where \mathcal{I} is a index set to label different orbits. For each Π_o ,

choose one of its element and denote it as $\pi_{o,e}$. The stablizer of each $\pi_{o,e}$ in G (*little group*) is denoted as S_o . There is a bijection from $G/S_o = \{hS_o\}$ to Π_o defined by

$$Ad_h(\pi_{o,e}) = h\pi_{o,e}h^{-1} =: \pi_{o,h} \quad (2)$$

where $\{h\}$ is a system of representatives of left cosets G/S_o . Define

$$\Pi_{o,h;\eta,i} \equiv \pi_{o,h}h \otimes_{S_o} e_{\eta,i}^o \quad (3)$$

in which $\{e_{\eta,i}^o | i = 1, 2, \dots, d_\eta^o\}$ with fixed o, η is the η th irreducible representation of S_o whose dimension is d_η^o , then

Proposition 1 (*little group method*)[11][12][13]

1. For each fixed (o, η) , $\{\Pi_{o,h;\eta,i}\}$ induces an irreducible representation of E , denoted as $D_{o,\eta}$;
2. If $(o, \eta) \neq (o', \eta')$, then $D_{o,\eta}$ and $D_{o',\eta'}$ are inequivalent;
3. $\{D_{o,\eta}\}$ gives all inequivalent irreducible representations of E .

II.2 Cubic group in any dimension

The symmetry group of a cube including inversions in three dimensional Euclidean space, which is denoted as O_h in the theory of point groups [15], can be generalized into any n -dimensional Euclidean space E^n , along two different approaches whose results are equivalent. The first approach of generalization which is very natural and straightforward is geometrical. An n -cube (or *hyper-cube* in E^n) C_n is defined to be a subset of E^n , $C_n = \{p | x^i(p) = \pm 1\}$, where $x^i : E^n \rightarrow \mathcal{R}, i = 1, 2, \dots, n$ are coordinate functions of E^n , together with the distance inherited from E^n . n -Cubic group (*hyper-cubic group of degree n*) O_n consists of all isometries of E^n which stabilize C_n . While the second approach of generalization is algebraic. O_h has a semi-direct product structure as $Z_2^3 \rtimes S_3$ [11]; we generalized this to $Z_2^n \rtimes S_n$ which is just a *wreath product* $Z_2 \wr S_n$ of Z_2 with S_n . We point out that these two generalizations are identical. Let $\{e_i\}$ be a standard orthogonal basis of E^n , namely $x^j(e_i) = \delta_i^j$. Define n points in C_n to be $p_0 = (-1, -1, \dots, -1), p_i = p_0 + 2e_i$.

Lemma 1 $\forall \epsilon \in O_n$, ϵ is entirely determined by images $\epsilon(p_i), i = 0, 1, 2, \dots, n$.

Proof:

The fact that ϵ is an isometry of E^n ensures the equality of Euclidean distances $d(p, p_i) = d(\epsilon(p), \epsilon(p_i))$, $i = 0, 1, \dots, n$ for any other p in C_n . If all $\epsilon(p_i)$ are given, $\epsilon(p)$ will be fixed for any other p accordingly due to the fundamental lemma of Euclidean geometry (lemma A-1 in Appendix A). In fact, the existence of solution in lemma A-1 is guaranteed by that ϵ stabilizes C_n and lemma A-1 itself ensures the uniqueness.

□

To fix $\epsilon(p_0)$, there are 2^n ways; while for a fixed $\epsilon(p_0)$, there are $n!$ possibilities to fix $\epsilon(p_i), i = 1, 2, \dots, n$. Therefore, $|O_n| = 2^n \cdot n!$.

Proposition 2 (*Structure of O_n*)

$$O_n \cong Z_2^n \rtimes S_n \quad (4)$$

Proof:

Introduce a class of isometries in E^n :

$$\sigma(e_i) = e_{\sigma(i)}; I_i(e_j) = (1 - 2\delta_{ij})e_j, i = 1, 2, \dots, n \quad (5)$$

where $\sigma \in S_n$ permute the axes and I_i inverts the i th axis. Subjected to the relations

$$I_i^2 = e, I_i I_j = I_j I_i, i, j = 1, 2, \dots, n; \sigma I_i = I_{\sigma(i)} \sigma, \sigma \in S_n \quad (6)$$

these isometries generate a sub-group of C_n isomorphic to $Z_2^n \rtimes S_n$ whose order is $2^n n! = (|O_n|)$.

So (4) follows.

□

A. Kerber gave a detailed introduction on the conjugate classification and representation theory of a general wreath product $N \wr G$ in [5]. We specify his general results to our case $Z_2^n \rtimes S_n \cong Z_2 \wr S_n$.

Some fundamental facts about symmetrical groups S_n should be recalled [15]. Each element $\sigma \in S_n$

has a cycle decomposition

$$\sigma = \begin{pmatrix} 1 & 2 & \dots & n \\ \sigma(1) & \sigma(2) & \dots & \sigma(n) \end{pmatrix} = \prod_{k=1}^n \prod_{\alpha=1}^{\nu_k} \tau_{k\alpha} \quad (7)$$

where $\tau_{k\alpha}$ are independent k -cycles, which can be expressed as $(a_1 a_2 \dots a_k)$, and write $n(k, \alpha) = \{a_1, a_2, \dots, a_n\}$. The cycle structure of σ can be represented formally as

$$(\nu) = \prod_{k=1}^n (k^{\nu_k}) \quad (8)$$

where $\{\nu_k\}$ satisfies $\sum_{k=1}^n k \cdot \nu_k = n$. Two elements in S_n are conjugate equivalent, iff they have the same cycle structure. The number of elements in class (ν) is equal to $N_{(\nu)} = n! / \prod_{k=1}^n (k^{\nu_k} \nu_k!)$. Each cycle structure (ν) can be visualized by one unique Young diagram which is denoted also by (ν) . There is a one-one correspondence between all inequivalent irreducible representations of S_n and all Young diagrams, which enable us to represent each irreducible representation by the corresponding Young diagram (ν) . We write the basis of one of these representations (ν) in $d_{(\nu)}$ dimension as $e_{(\nu)i}, i = 1, 2, \dots, d_{(\nu)}$.

We point out that the conjugate classification of O_n has a deep relation with that of S_n . A generic element in $Z_2 \wr S_n$ can be written as

$$\sigma \cdot \prod_i I_i^{s_i} = \begin{pmatrix} 1 & 2 & \dots & n \\ (-)^{s_1} \sigma(1) & (-)^{s_2} \sigma(2) & \dots & (-)^{s_n} \sigma(n) \end{pmatrix} \quad (9)$$

in which $s_i \in \mathbb{Z}/2\mathbb{Z}$. We call the r.h.s. of (9) by *permutation with signature*. $\sigma \prod_i I_i^{s_i}$ can be decomposed according to (7), i.e. $\prod_i I_i^{s_i} = \prod_{k=1}^n \prod_{\alpha=1}^{\nu_i} \prod_{a \in n(k, \alpha)} I_a^{s_a}$ and

$$\sigma \prod_i I_i^{s_i} = \prod_{k=1}^n \prod_{\alpha=1}^{\nu_i} (\tau_{k\alpha} \prod_{a \in n(k, \alpha)} I_a^{s_a}) \quad (10)$$

The *cycle with signature* is defined to be

$$\tau_{k\alpha} \prod_{a \in n(k, \alpha)} I_a^{s_a} = \begin{pmatrix} a_1 & a_2 & \dots & a_k \\ (-)^{s_{a_1}} a_2 & (-)^{s_{a_2}} a_3 & \dots & (-)^{s_{a_k}} a_1 \end{pmatrix}$$

For two independent $(k, \alpha), (k', \alpha')$, it is easy to verify that

$$\tau_{k\alpha}\tau_{k'\alpha'} = \tau_{k'\alpha'}\tau_{k\alpha}, \quad \prod_{a \in n(k, \alpha)} I_a^{s_a} \tau_{k'\alpha'} = \tau_{k'\alpha'} \prod_{a \in n(k, \alpha)} I_a^{s_a}, \quad \prod_{a \in n(k', \alpha')} I_a^{s_a} \tau_{k\alpha} = \tau_{k\alpha} \prod_{a \in n(k', \alpha')} I_a^{s_a}$$

Proposition 3 [3][4][5] *We use \sim to denote conjugate equivalent.*

1. (descent rule)

$$\sigma \prod_i I_i^{s_i} \sim \sigma' \prod_i I_i^{s'_i} \Rightarrow \sigma \sim \sigma' \quad (11)$$

2. (permutation rule) Let

$$\tilde{\sigma} = \begin{pmatrix} 1 & 2 & \dots & n \\ \tilde{\sigma}(1) & \tilde{\sigma}(2) & \dots & \tilde{\sigma}(n) \end{pmatrix} = \begin{pmatrix} \sigma(1) & \sigma(2) & \dots & \sigma(n) \\ \sigma'(1) & \sigma'(2) & \dots & \sigma'(n) \end{pmatrix}$$

then

$$\tilde{\sigma}(\sigma \prod_i I_i^{s_i})\tilde{\sigma}^{-1} = \begin{pmatrix} \tilde{\sigma}(1) & \tilde{\sigma}(2) & \dots & \tilde{\sigma}(n) \\ (-)^{s_1}\sigma'(1) & (-)^{s_2}\sigma'(2) & \dots & (-)^{s_n}\sigma'(n) \end{pmatrix} \quad (12)$$

3. (signature rule within one cycle) Let $\tau_{k\alpha}$ be a k -cycle and a_0 be a given number in $n(k, \alpha)$, then

$$\tau_{k\alpha} \prod_{a \in n(k, \alpha)} I_a^{s_a} \sim \tau_{k\alpha} \prod_{a \in n(k, \alpha)} I_a^{s_a + \delta_{aa_0} + \delta_{a, \tau_{k\alpha}(a_0)}} \quad (13)$$

Note that $\tau_{k\alpha}(a_0)$ is calculated modulo k (the subscripts of I_a are always understood in this way).

4. (signature rule between two cycles) Let $\tau_{k\alpha}, \tau_{k\beta}$ be two independent k -cycles and we define a bijection $\theta : n(k, \alpha) \rightarrow n(k, \beta), a_i \mapsto b_i$. Then

$$\tau_{k\alpha} \prod_{a \in n(k, \alpha)} I_a^{s_a} \cdot \tau_{k\beta} \prod_{b \in n(k, \beta)} I_b^{s_b} \sim \tau_{k\alpha} \prod_{a \in n(k, \alpha)} I_a^{s_{\theta(a)}} \cdot \tau_{k\beta} \prod_{b \in n(k, \beta)} I_b^{s_{\theta^{-1}(b)}} \quad (14)$$

This theorem ensures conjugate classification of $Z_2 \wr S_n$ is totally determined by the structure of cycles with signature. We verify this statement by generalizing Young diagram technology. First, draw a *Young diagram with numbers and signatures* for each element $\sigma \prod_i I_i^{s_i} \in Z_2 \wr S_n$ according to the decomposition Eq.(10) by the following rules:

1. Plot Young diagram of the class in S_n to which σ belongs and fill each row of this Young diagram with numbers in corresponding cycle by cyclic ordering from up-most box to down-most box.
2. Draw a *slash* in the Young box if the number in this box is mapped to a minus-signed number.

Secondly, partition elements in $Z_2 \wr S_n$ by their cycle structure in S_n and Eq.(11) guarantees elements belong to different partitions can not be conjugate equivalent. Eq.(12) implies that all the numbers that we filled by rule 1 are unnecessary, so smear them out and leave boxes and *slashes* only. Within each row, Eq.(13) says that the positions of *slashes* make no difference. What's more, in fact only that the total number of *slashes* is even or odd distinguishes different classes. Therefore we regulate each row to contain zero or one *slash* at the bottom box. Eq.(14) shows that we can not distinguish the case that one row without any *slash* (Mr. Zero) is put to the left to another row with one *slash* (Mr. One) from that Mr.Zero is to the right of Mr.One, if they have same cyclic length; thus we regulate that Mr.Zero shall always stand left to Mr.One. Therefore, conjugate classes of $Z_2 \wr S_n$ can be uniquely characterized by generalizing Young diagrams containing *slashes*. Following Eq.(8), we represent conjugate classes by

$$(\nu^+, \nu^-) = \prod_{k=1}^n (k^{\nu_k^+ + \nu_k^-}) \quad (15)$$

where ν_k^+ is the number of Mr.Zero-type k -cycles and ν_k^- is that of Mr.One-type k -cycles, which satisfy $\nu_k^+ + \nu_k^- = \nu_k$. It is not difficult to check some numerical properties of conjugate classes of $Z_2 \wr S_n$.

Corollary 1 1. Given a class (ν) in S_n , there are

$$\prod_{k=1}^n (1 + \nu_k) \quad (16)$$

classes in $Z_2 \wr S_n$ which descend to (ν) .

2. The number of elements in a class (ν^+, ν^-) is

$$N_{(\nu^+, \nu^-)} = N_{(\nu)} \prod_{k=1}^n (C_{\nu_k^+}^{\nu_k^+} (\sum_{i=0}^{\lfloor \frac{k}{2} \rfloor} C_k^{2i})^{\nu_k^+} (\sum_{j=1}^{\lfloor \frac{k+1}{2} \rfloor} C_k^{2j-1})^{\nu_k^-}) \quad (17)$$

where C_m^n is combinatorial number defined to be $m!/(n!(m-n)!)$.

3. The order of a class (ν^+, ν^-) is

$$lcm(\{k \cdot 2^{\delta(\nu_k^-)} | \nu_k \neq 0\}) \quad (18)$$

where $\delta(\nu_k^-) = 0$, if $\nu_k^- = 0$; $\delta(\nu_k^-) = 1$, if $\nu_k^- > 0$.

4. Determinant (signature, parity) of a class

$$det((\nu^+, \nu^-)) = (-1)^{\sum_{k=1}^n \nu_k^-} \cdot det((\nu)) \quad (19)$$

where $det((\nu))$ is the determinant of (ν) in S_n .

All inequivalent irreducible representations of Z_2^n can be expressed as

$$\chi_{(s)} := \bigotimes_{p=1}^n \chi_{(-)}^{s_p} \quad (20)$$

in which $s_p \in \mathbb{Z}/2\mathbb{Z}$, $p = 1, 2, \dots, n$ and $\chi_{(-)}, \chi_{(+)}$ are two irreducible representations of Z_2 with $\chi_{(+)}$ being the unit representation. Thus $\pi_{(s)}$ can be defined by Eq.(1) and $\Pi(Z_2^n) = \{\pi_{(s)}\}$. Note that $\pi_{(s)}$ satisfy $\pi_{(s)}\pi_{(s')} = \pi_{(s \cdot s')}$ where $(s \cdot s')(p) = s(p)s'(p)$. $\Pi(Z_2^n)$ is divided into $n+1$ orbits under the S_n -action, namely $\Pi(Z_2^n) = \bigsqcup_{p=0}^n \Pi_p$. For a given p , Π_p consists of those $\pi_{(s)}$ who has p components in (s) equal to 1, other $n-p$ components equal to 0; hence $|\Pi_p| = C_n^p$. Each $\pi_{p,e}$ is specified to a $\pi_{(s)}$ with $s_p = 0, p = 1, 2, \dots, n-p$; $s_p = 1, p = n-p+1, \dots, n$, whose stationary subgroup is just $S_{(n-p)} \otimes S_p$, denoted as F_p . Representatives of left-cosets in S_n/F_p are written as σ_r , then according to Eqs.(2)(3) and Theorem 1,

Proposition 4 (Representation theory of $Z_2 \wr S_n$)

$$\Pi_{p,\sigma_r;(\mu)i,(\nu)j} = \pi_{p,\sigma_r}\sigma_r \otimes_{F_p} (e_{(\mu)i} \otimes e_{(\nu)j})$$

give all inequivalent irreducible representations of $Z_2 \wr S_n$ when $(p, (\mu), (\nu))$ runs over its domain.

where $\pi_{p,\sigma_r} = Ad_{\sigma_r}(\pi_{p,e})$ whose (s) will be denoted as $(s^{(p\sigma_r)})$.

Corollary 2 1. (Burnside formula) $\sum_{(p,(\mu),(\nu))} (C_n^p d_{(\mu)} d_{(\nu)})^2 = 2^n n!$

2. The number of conjugate classes is $\sum_{(p,(\mu),(\nu))} 1$.

3. (Representation matrix element) Given $\sigma \prod_q I_q^{t_q} \in Z_2 \wr S_n$,

$$D_{(p,(\mu),(\nu))}(\sigma \prod_q I_q^{t_q})_{\sigma_r i j}^{\sigma'_r i' j'} = \delta_{\tilde{\sigma}_r(\sigma \sigma_r)}^{\sigma'_r} D_{(\mu)}(\sigma_{(n-p)}(\sigma \sigma_r))_i^{i'} D_{(\nu)}(\sigma_p(\sigma \sigma_r))_j^{j'} \prod_q (-)^{s_q^{(p \sigma_r)} t_q}$$

4. (Character)

$$\chi_{(p,(\mu),(\nu))}(\sigma \prod_q I_q^{t_q}) = \delta_{\tilde{\sigma}_r(\sigma \sigma_r)}^{\sigma'_r} \chi_{(\mu)}(\sigma_{(n-p)}(\sigma \sigma_r)) \chi_{(\nu)}(\sigma_p(\sigma \sigma_r)) \prod_q (-)^{s_q^{(p \sigma_r)} t_q}$$

where $\tilde{\sigma}_r, \sigma_{(n-p)}, \sigma_p$ map an element in S_n to its decompositions according to S_n/F_p , S_{n-p} and S_p respectively.

In the end of this subsection, we introduce the *orientation-preserved* n-cubic group SO_n which is a normal subgroup of O_n

$$SO_n := (O_n \cap SO(n)) \triangleleft O_n \quad (21)$$

Define $Z_2^n|_e$ as a subgroup of Z_2^n generated by $I_i I_j, i \neq j$ and $Z_2^n|_o := Z_2^n \setminus Z_2^n|_e$. Then

$$SO_n = (Z_2^n|_e \cdot A_n) \bigsqcup (Z_2^n|_o \cdot (S_n \setminus A_n)) \quad (22)$$

in which \cdot is product of two subsets in a group, A_n stands for the alternative subgroup in S_n . Thus, $|SO_n| = (2^n \cdot (n!))/2$.

II.3 Double group and spinor representation

Some fundamental facts of Clifford algebra are necessary for giving the definition and properties of double groups. Denote the Clifford algebra upon Euclidean space V as $Cl(V)$; the isometry $x \mapsto -x$ on V extends to an automorphism of $Cl(V)$ denoted by $x \mapsto \tilde{x}$ and referred to as the canonical automorphism of $Cl(V)$. We use $Cl^*(V)$ to denote the multiplicative group of invertible elements in $Cl(V)$ and the Pin group is the subgroup of $Cl^*(V)$ generated by unit vectors in V , i.e.

$$Pin(V) := \{a \in Cl^*(V) : a = u_1 \cdots u_r, u_j \in V, \|u_j\| = 1\}$$

Proofs of the following four statements can be found in [16].

Lemma 2 *If $u \in V$ is nonnull, then R_u , reflection along u , is given in terms of Clifford multiplication by*

$$R_u x = -u x u^{-1}, \forall x \in V$$

Theorem 2 *The sequence*

$$0 \rightarrow Z_2 \rightarrow Pin(V) \xrightarrow{\widetilde{Ad}} O(V) \rightarrow 1$$

is exact, in which

$$\widetilde{Ad}_a(x) := \tilde{a} x a^{-1}, \forall x \in Cl(V), a \in Pin(V)$$

We will usually write \widetilde{Ad} just by π as a surjective homomorphism.

Proposition 5 *$Cl(E^4)$, as an associative algebra with unit, is isomorphic to $M_2(\mathbf{H})$ where \mathbf{H} denotes quaternions.*

Lemma 3 *Under the above algebra isomorphism, the image of $Pin(E^4)$ is a subset of $SU(4)$.*

Now we give the main definition of this paper.

Definition 1 *Let ϵ be an injective homomorphism from a group G to $O(n)$, then the double group or the spin-extension of G with respect to ϵ is defined to be $D_n(G, \epsilon) := \pi^{-1}(G)$.*

An introduction to double groups in three dimension can be found in [17]. Following elementary facts in the theory of group extension [18], this diagram

$$\begin{array}{ccccccc} 0 & \longrightarrow & Z_2 & \longrightarrow & Pin(E^n) & \xrightarrow{\pi} & O(n) \longrightarrow 1 \\ & & \parallel & & \uparrow & & \uparrow \epsilon \\ 0 & \longrightarrow & Z_2 & \longrightarrow & \pi^{-1}(G) & \longrightarrow & G \longrightarrow 1 \end{array}$$

is commutative. If $(G, \epsilon_1) \sim (G, \epsilon)$, there is

$$\begin{array}{ccccccc} 0 & \longrightarrow & Z_2 & \xrightarrow{i} & \pi^{-1}((G_1, \epsilon_1)) & \xrightarrow{\pi} & (G, \epsilon_1) \longrightarrow 1 \\ & & \parallel & & \downarrow & & \downarrow \\ 0 & \longrightarrow & Z_2 & \xrightarrow{i'} & \pi^{-1}((G_2, \epsilon_2)) & \xrightarrow{\pi'} & (G, \epsilon_2) \longrightarrow 1 \end{array}$$

Note that the double group is not a universal object for a given abstract group G but a special type of Z_2 -central extension of G subjected to the embedding ϵ . For example, the results of doubling two Z_2 subgroups in $O(2)$, $I := \{1, \sigma\}$, $R := \{1, R(\pi)\}$ where σ denotes reflection along y -axes and $R(\pi)$ is the rotation over π , are $\pi^{-1}(I) \cong Z_2 \otimes Z_2$ while $\pi^{-1}(R) \cong Z_4$. Nevertheless, we will use symbol \bar{G} to denote the double group at most cases where n and ϵ are fixed. Meanwhile, symbol \bar{e} is adopted to refer -1 in Clifford algebra and is called *central element*.

Let $s : G \rightarrow \bar{G}$, s.t. $\pi s = Id_G$, namely s is a cross-section of π . There is a property of the conjugate classes of \bar{G} which is easy to verify.

Lemma 4 *Let C be a conjugate class in G , then either will $\pi^{-1}(C)$ be one conjugate class in \bar{G} satisfying $\forall g \in C, s(g) \sim -s(g)$; or it will split into two conjugate classes C_1, C_2 in \bar{G} s.t. $\forall g \in C, s(g) \in C_1 \Leftrightarrow -s(g) \in C_2$.*

We will give a more deep result on the splitting of conjugate classes when doubling G to \bar{G} in our another paper.

Let r be an irreducible representation of \bar{G} on V , then $r(-1) = \pm \mathbf{1}$.

Definition 2 *An irreducible representation of \bar{G} with $r(-1) = \mathbf{1}$ is called a single-valued representation of G while an irreducible representation with $r(-1) = -\mathbf{1}$ is called a spinor representation or two-valued representation of G .*

Proposition 6 *Let $IRR_C(G)$ be the class of all inequivalent irreducible representations of G and $IRR_C(G)^s$ be the class of all inequivalent single-valued representations of G , define $\phi : IRR_C(G) \rightarrow IRR_C(G)^s, r \mapsto r \circ \pi$. Then ϕ is a bijection.*

Proof:

One can check: $r \circ \pi$ is a representation of \bar{G} ; if $r \cong r'$, then $r \circ \pi \cong r' \circ \pi$; that r is irreducible implies that $r \circ \pi$ is irreducible and $r \circ \pi$ is single-valued. Therefore, ϕ is well-defined. If r and r'

are inequivalent, then $r \circ \pi$ and $r' \circ \pi$ are two elements in $IRR_{\mathcal{C}}(G)^s$, namely ϕ is injective. To prove that ϕ is a surjection, consider any $\tilde{r} \in IRR_{\mathcal{C}}(G)^s : \bar{G} \rightarrow V$. Define $r : G \rightarrow V, g \mapsto \tilde{r}(s(g))$ where $s(g)$ is any element in $\pi^{-1}(g)$. One can check: r is a well-defined map since \tilde{r} is single-valued; r is an irreducible representation of G on V , accordingly $r \in IRR_{\mathcal{C}}(G)$ and lastly, $\phi(r) = \tilde{r}$. So the result follows.

□

This proposition says that all single-valued representations of G which are part of inequivalent irreducible representations of \bar{G} are completely determined by the representation theory of G .

III Structure of $\overline{O_4}$

III.1 Structure of O_4

Following Proposition 2, $O_4 \cong Z_2^4 \rtimes S_4$; hence $|O_4| = 384$. In point group theory, rotation subgroup of O_h is denoted as O and $S_4 \cong Z_2^2 \rtimes S_3 \cong O$ [11]. We write the isomorphism explicitly. The structure of $Z_2^2 \rtimes S_3$ is given by four generators α, β, η, t and the relations

$$\alpha^2 = e, \beta^2 = e, \alpha\beta = \beta\alpha \quad (23)$$

$$t^3 = e, \eta^2 = e, \eta t = t^2 \eta \quad (24)$$

$$t\alpha = \alpha\beta t, t\beta = \alpha t, \eta\alpha = \beta\eta \quad (25)$$

and the isomorphisms are defined to be

$$(12)(34) \leftrightarrow \alpha \leftrightarrow \text{diag}(-1, -1, 1), (13)(24) \leftrightarrow \beta \leftrightarrow \text{diag}(1, -1, -1)$$

$$(234) \leftrightarrow t \leftrightarrow \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix}, (23) \leftrightarrow \eta \leftrightarrow \begin{pmatrix} 0 & 0 & -1 \\ 0 & -1 & 0 \\ -1 & 0 & 0 \end{pmatrix}$$

The structure of $Z_2^4 \rtimes S_4$ is given by (23)(24)(25) together with (see Eq.(6))

$$I_i^2 = e, I_i I_j = I_j I_i, i, j = 1..4, i \neq j \quad (26)$$

$$\begin{aligned}
\alpha I_1 &= I_2 \alpha, \alpha I_3 = I_4 \alpha \\
\beta I_1 &= I_3 \beta, \beta I_2 = I_4 \beta \\
t I_1 &= I_1 t, t I_2 = I_4 t, t I_3 = I_2 t, t I_4 = I_3 t \\
\eta I_1 &= I_1 \eta, \eta I_2 = I_3 \eta, \eta I_4 = I_4 \eta
\end{aligned} \tag{27}$$

The matrix form of O_4 is given by (see Eq.(5))

$$(I_I)_k^j = \delta_k^j (1 - 2\delta_i^j), i, j, k = 1, 2, 3, 4 \tag{28}$$

$$\alpha \mapsto \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix} \beta \mapsto \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix} t \mapsto \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \end{pmatrix} \eta \mapsto \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \tag{29}$$

In fact, if we introduce

$$\gamma \mapsto \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

then the generating relations of O_4 can be reduced to a smaller set $\{I_i, \gamma, t | i = 1, 2, 3, 4\}$ whose generating relations are (26)(27) together with

$$\gamma^2 = e, t^3 = e, (t\gamma)^4 = e \tag{30}$$

$$\gamma I_1 = I_3 \gamma, \gamma I_2 = I_2 \gamma, \gamma I_4 = I_4 \gamma \tag{31}$$

while $\alpha = (t^2 \gamma)^2, \beta = t \gamma t^2 \gamma t, \eta = \gamma t \gamma t^2 \gamma$.

Applying the general results on conjugate classification of O_n Eqs.(15)(16)...(19), we give the table of conjugate classes of O_4 (see Tab. I).

III.2 Construction of \overline{O}_4

We will denote $s(g)$ still as g for all $g \in G$. \overline{O}_4 is generated by the equations below.

Proposition 7

$$I_i^2 = -1, I_i I_j = -I_j I_i, i, j = 1..4, i \neq j \quad (32)$$

$$\gamma^2 = -1, t^3 = -1, (t\gamma)^4 = -1 \quad (33)$$

$$\underline{\gamma I_1 = -I_3 \gamma}, \gamma I_2 = -I_2 \gamma, \gamma I_4 = -I_4 \gamma \quad (34)$$

$$t I_1 = I_1 t, \underline{t I_2 = I_4 t}, \underline{t I_3 = I_2 t}, \underline{t I_4 = I_3 t} \quad (35)$$

Proof:

First Eqs.(32)...(35) are valid. In fact, the standard orthogonal bases in E^4 satisfy Clifford relations $e_i e_j + e_j e_i = -2\delta_{ij}$ which is equivalent to (32); therefore, one can take $I_i = e_i$. Following lemma 2, we set $\gamma = \frac{1}{\sqrt{2}}(e_3 - e_1)$ and check that (34) is satisfied. Let $t = \frac{1}{2}(1 - e_2 e_3 + e_2 e_4 - e_3 e_4)$ which is the product of $\frac{1}{\sqrt{2}}(e_2 - e_3)$ and $\frac{1}{\sqrt{2}}(e_4 - e_2)$, and (35) can be verified. Finally, one can check that (33) is also satisfied.

Second, notice that above equations are just Eqs.(26)(27)(30)(31), which generate O_4 , twisted by a Z_2 factor set. So due to the validity of the above equations, $\forall g \in \overline{O_4}$, either g or $-g$ will be generated. But -1 can be generated. Therefore, the above equation set generations $\overline{O_4}$.

□

We can give another proof of this result by proposition 5. In fact, we introduce γ -matrices in E^4 as

$$\gamma_i = \begin{pmatrix} \mathbf{0}_{2 \times 2} & i\sigma_i \\ i\sigma_i & \mathbf{0}_{2 \times 2} \end{pmatrix}, i = 1, 2, 3; \gamma_4 = \begin{pmatrix} \mathbf{0}_{2 \times 2} & -\mathbf{1}_{2 \times 2} \\ \mathbf{1}_{2 \times 2} & \mathbf{0}_{2 \times 2} \end{pmatrix}$$

in which σ_i s stand for three Pauli matrices

$$\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \sigma_2 = \begin{pmatrix} 0 & i \\ -i & 0 \end{pmatrix}, \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

Note that our convention here has some difference with the usual one in physics. $\gamma_i (i = 1..4)$ satisfy Clifford relations $\gamma_i \gamma_j + \gamma_j \gamma_i = -2\delta_{ij} \mathbf{1}_{4 \times 4}$ and $\gamma_i^\dagger = -\gamma_i, \gamma_i \gamma_i^\dagger = \mathbf{1}_{4 \times 4}, \det(\gamma_i) = 1$.

We use $S(g)$ as the image of $s(g)$ in $M_2(\mathbf{H})$. Let

$$S(I_i) = \gamma_i, S(\gamma) = \frac{i}{\sqrt{2}} \cdot \begin{pmatrix} 0 & 0 & 1 & -1 \\ 0 & 0 & -1 & -1 \\ 1 & -1 & 0 & 0 \\ -1 & -1 & 0 & 0 \end{pmatrix}, S(t) = \frac{e^{i\frac{7\pi}{4}}}{\sqrt{2}} \cdot \begin{pmatrix} 1 & -i & 0 & 0 \\ 1 & i & 0 & 0 \\ 0 & 0 & i & 1 \\ 0 & 0 & -i & 1 \end{pmatrix} \quad (36)$$

then one can check that these matrices give correct images under \widetilde{Ad} and satisfy the corresponding relations in (32)...(35). It should be noticed that the \widetilde{Ad} -map condition can fix these matrices up to a non-vanishing scalar and that by using lemma 3, the scalar can be fixed up to a Z_4 uncertainty, namely if one find out a $S(g)$ then $iS(g)$, $-S(g)$, $-iS(g)$ will also work. Then one can figure out two of them by calculating the projections on the basis of $Cl(E^4)$ and ruling out those whose projections are pure imagine.

We point out that the generating relations in proposition 7 are not unique, due to the canonical automorphism of $Cl(E^4)$. In fact from the second proof of this proposition, we have notified that at last there is still a Z_2 uncertainty. Consequently, we can change the cross-section s to another one s' by a “local” Z_2 transformation and the underlined equations in Eqs.(32)...(35) may gain or lose some -1 -factors accordingly. Anyway, they are equivalent to the former ones.

To classify the elements in \overline{O}_4 , Lemma 4 will enable us to use the same symbols for the conjugate classes of O_4 and to use a “ ν ” for those splitting classes. Except for classes 1, 8, 14, 15, 20 which split into two classes for each, any other class in O_4 is lifted to one class. Therefore, there are totally 25 classes in \overline{O}_4 (see table II).

IV Representations of \overline{O}_4

IV.1 Single-valued representations of O_4

Due to Theorem 6, there are totally 20 inequivalent single-valued representations of O_4 corresponding to the 20 inequivalent irreducible representations of O_4 ; the representation theory of O_4 can be systematically solved by applying little group method (Theorem 1).

All inequivalent irreducible characters of Z_2^4 are listed in Table III. Following Theorem 4, $\Pi(Z_2^4)$ are partitioned into orbits with index set defined in a physical convention $\mathcal{I} := \{S, P, V, A, T\}$.

$$\begin{aligned}\Pi_S &= \{\pi_{0000}\}, F_S \cong S_4; \Pi_P = \{\pi_{1111}\}, F_P \cong S_4; \\ \Pi_V &= \{\pi_{0001}, \pi_{0010}, \pi_{0100}, \pi_{1000}\}, F_V \cong S_3; \\ \Pi_A &= \{\pi_{1110}, \pi_{1101}, \pi_{1011}, \pi_{0111}\}, F_A \cong S_3; \\ \Pi_T &= \{\pi_{0011}, \pi_{0101}, \pi_{1001}, \pi_{0110}, \pi_{1010}, \pi_{1100}\}, F_T \cong Z_2^2\end{aligned}$$

We will use $[\lambda]$ instead of (ν) to denote Young diagrams where $[\lambda] = [\lambda_1 \lambda_2 \dots \lambda_n]$, $\lambda_k = \sum_{i=k}^n \nu_i$.

Orbit S

All inequivalent irreducible representations of S_4 is labeled by $[4], [31], [2^2], [21^2], [1^4]$; accordingly,

$$\Pi_S \cdot ([4], [31], [2^2], [21^2], [1^4])$$

provide two one-dimensional, one two-dimensional and two three-dimensional representations. As for representation matrices, all $I_i, i = 1..4$ are mapped to identity, while α, β, t, η take the same matrix form as they have in S_4 , i.e. $\Pi_S \cdot [4] : I_i, \alpha, \beta, t, \eta \rightarrow 1$; $\Pi_S \cdot [1^4] : I_i, \alpha, \beta, t \rightarrow 1, \eta \rightarrow -1$;

$$\Pi_S \cdot [2^2] : I_i, \alpha, \beta \rightarrow \mathbf{1}_{2 \times 2}, t \rightarrow \begin{pmatrix} \omega & 0 \\ 0 & \omega^2 \end{pmatrix}, \eta \rightarrow \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix};$$

$$\Pi_S \cdot [31] : I_i \rightarrow \mathbf{1}_{3 \times 3}, \alpha \rightarrow \begin{pmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix}, \beta \rightarrow \begin{pmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, t \rightarrow \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}, \eta \rightarrow \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix};$$

$\Pi_S \cdot [21^2]$: I_i, α, β, t take the same form of $\Pi_S \cdot [31]$ and η gains a minus sign compared to $\Pi_S \cdot [31]$.

Orbit P

$$\Pi_P \cdot ([4], [31], [2^2], [21^2], [1^4])$$

The only difference from **Orbit S** is that I_i are mapped to $-\mathbf{1}$.

Orbit V

All inequivalent irreducible representations of S_3 can be written as $[3], [21], [1^3]$ and it has no difficulty, using our generating relations, to check

$$\alpha\pi_{1000}\alpha^{-1} = \pi_{0100}, \eta\pi_{0100}\eta^{-1} = \pi_{0010}, \alpha\pi_{0010}\alpha^{-1} = \pi_{0001}$$

Hence, this orbit gives two four-dimensional representations and one eight-dimensional representation.

$$\Pi_V \cdot ([3], [21], [1^3]) = (e, \alpha, \beta\eta, \alpha\beta\eta) \cdot \pi_{1000} \cdot ([3], [21], [1^3])$$

The representation matrices of $\Pi_V \cdot [3]$ are coincident with those in Eqs.(28)(29). Representation matrices of $\Pi_V \cdot [1^3]$ are the same as those in $\Pi_V \cdot [3]$, except that η picking on a minus sign.

$\Pi_V \cdot [21]$:

$$e_i \rightarrow \begin{pmatrix} \Pi_V \cdot [3](e_i) & \mathbf{0}_{4 \times 4} \\ \mathbf{0}_{4 \times 4} & \Pi_V \cdot [3](e_i) \end{pmatrix},$$

$$\alpha \rightarrow \begin{pmatrix} \Pi_V \cdot [3](\alpha) & \mathbf{0}_{4 \times 4} \\ \mathbf{0}_{4 \times 4} & \Pi_V \cdot [3](\alpha) \end{pmatrix}, \beta \rightarrow \begin{pmatrix} \mathbf{0}_{4 \times 4} & \Pi_V \cdot [3](\beta) \\ \Pi_V \cdot [3](\beta) & \mathbf{0}_{4 \times 4} \end{pmatrix},$$

$$t \rightarrow \begin{pmatrix} \omega \cdot \Pi_V \cdot [3](t) & \mathbf{0}_{4 \times 4} \\ \mathbf{0}_{4 \times 4} & \omega^2 \cdot \Pi_V \cdot [3](t) \end{pmatrix}, \eta \rightarrow \begin{pmatrix} \mathbf{0}_{4 \times 4} & \Pi_V \cdot [3](\eta) \\ \Pi_V \cdot [3](\eta) & \mathbf{0}_{4 \times 4} \end{pmatrix}$$

Orbit A

Similar to **Orbit V**, there are two four-dimensional representations and one eight-dimensional representation.

$$\Pi_A \cdot ([3], [21], [1^3]) = (e, \alpha, \beta\eta, \alpha\beta\eta) \cdot \pi_{0111} \cdot ([3], [21], [1^3])$$

while the representation matrices for I_i pick on a minus sign, without changing the others.

Orbit T

The stationary subgroup F_T leaving π_{0110} invariant is $\{e, \eta, \alpha\beta, \alpha\beta\eta\}$ with four one-dimensional irreducible representations, denoted by $\pi_{(a,b)}$, $a, b = 0, 1$. Therefore, there are four six-dimensional representations given by this orbit. Notice that

$$\begin{aligned} \alpha\pi_{0110}\alpha^{-1} &= \pi_{1001}, t\pi_{1001}t^{-1} = \pi_{1010}, t\pi_{1010}t^{-1} = \pi_{1100}, \\ \alpha\pi_{1010}\alpha^{-1} &= \pi_{0101}, \beta\pi_{1100}\beta^{-1} = \pi_{0011}, \end{aligned}$$

the four representations can be labeled as

$$\Pi_T \cdot (\pi_{00}, \pi_{01}, \pi_{10}, \pi_{11}) = (e, \alpha, \alpha\beta t, \beta t^2, \beta t, t^2) \cdot \pi_{0110} \cdot (\pi_{00} + \pi_{01} + \pi_{10} + \pi_{11})$$

Then we enumerate the matrices for the four representations.

$$\Pi_T \cdot \pi_{00} : I_1 \rightarrow \text{diag}(1, -1, -1, -1, 1, 1), I_2 \rightarrow \text{diag}(-1, 1, 1, -1, -1, 1),$$

$$I_3 \rightarrow \text{diag}(-1, 1, -1, 1, 1, -1), I_4 \rightarrow \text{diag}(1, -1, 1, 1, -1, -1),$$

$$\alpha \rightarrow \begin{pmatrix} 0 & 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix} \beta \rightarrow \begin{pmatrix} 0 & 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \end{pmatrix}$$

$$t \rightarrow \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \end{pmatrix} \quad \eta \rightarrow \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 & 0 \end{pmatrix}$$

$$\Pi_T \cdot \pi_{01} : I_1 \rightarrow \Pi_T \cdot \pi_{01}(I_1), I_2 \rightarrow \Pi_T \cdot \pi_{01}(I_2), I_3 \rightarrow \Pi_T \cdot \pi_{01}(I_3), I_4 \rightarrow \Pi_T \cdot \pi_{01}(I_4)$$

$$\alpha \rightarrow \begin{pmatrix} 0 & 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -1 \end{pmatrix} \quad \beta \rightarrow \begin{pmatrix} 0 & -1 & 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \end{pmatrix}$$

$$t \rightarrow \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -1 & 0 \end{pmatrix} \quad \eta \rightarrow \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 & -1 & 0 \end{pmatrix}$$

$$\Pi_T \cdot \pi_{10} : I_1 \rightarrow \Pi_T \cdot \pi_{10}(I_1), I_2 \rightarrow \Pi_T \cdot \pi_{10}(I_2), I_3 \rightarrow \Pi_T \cdot \pi_{10}(I_3), I_4 \rightarrow \Pi_T \cdot \pi_{10}(I_4)$$

$$\alpha \rightarrow \Pi_T \cdot \pi_{10}(\alpha), \beta \rightarrow \Pi_T \cdot \pi_{10}(\beta), t \rightarrow \Pi_T \cdot \pi_{10}(t), \eta \rightarrow (-1) \cdot \Pi_T \cdot \pi_{10}(\eta)$$

$$\Pi_T \cdot \pi_{11} : I_1 \rightarrow \Pi_T \cdot \pi_{11}(I_1), I_2 \rightarrow \Pi_T \cdot \pi_{11}(I_2), I_3 \rightarrow \Pi_T \cdot \pi_{11}(I_3), I_4 \rightarrow \Pi_T \cdot \pi_{11}(I_4)$$

$$\alpha \rightarrow \Pi_T \cdot \pi_{11}(\alpha), \beta \rightarrow \Pi_T \cdot \pi_{11}(\beta), t \rightarrow \Pi_T \cdot \pi_{11}(t), \eta \rightarrow (-1) \cdot \Pi_T \cdot \pi_{11}(\eta)$$

Here we find all 20 inequivalent irreducible representations corresponding to the 20 conjugate classes of O_4 , which satisfy Burside formula

$$2 \times (1^2 + 1^2 + 2^2 + 3^2 + 3^2) + 2 \times (4^2 + 4^2 + 8^2) + 4 \times 6^2 = 384$$

Following proposition 6, we have found all of the single-valued representations of O_4 .

IV.2 Spinor representations of O_4

Notice the following facts that $\overline{Z_2^4} \triangleleft \overline{O_4}$, $\overline{O_4}/\overline{Z_2^4} \cong S_4$ and Eqs.(36) generate a spinor representation of O_4 which is denoted still as S ; what's more, its restriction to $\overline{Z_2^4}$ is also a two-valued representation of Z_2^4 . These facts ensure two conditions in Theorem 1. To apply Theorem 1 to deduce spinor representations of O_4 , we develop a calculation method. The matrices of a spinor representation of O_4 for I_i, γ, t , denoted as $\tilde{S}(I_i), \tilde{S}(\gamma), \tilde{S}(t)$, can be decomposed as

$$\begin{aligned}\tilde{S}(I_i) &= S(I_i) \otimes \mathbf{1}, i = 1, 3, 4; \tilde{S}(I_2) = -S(I_2) \otimes \mathbf{1}; \\ \tilde{S}(\gamma) &= \Gamma \otimes \tilde{\gamma}; \tilde{S}(t) = T \otimes \tilde{t}\end{aligned}$$

where Γ, T and $S(I_i)$ act on the same module, $\tilde{\gamma}, \tilde{t}$ have *the same texture (zero matrix elements) of the representation matrices of five inequivalent irreducible representations of S_4* (the minus added before $S(I_2)$ is for a physical convention). There are five spinor representations of dimension 4,4,8,12 and 12 respectively and the second half of Burside formula is satisfied.

$$4^2 + 4^2 + 8^2 + 12^2 + 12^2 = 384$$

Corresponding the generating equations (32)...(35), there are a system of matrix equations.

$$\tilde{S}(\gamma)^2 = \tilde{S}(t)^3 = -\mathbf{1}, (\tilde{S}(\gamma)\tilde{S}(t))^4 = -\mathbf{1} \quad (37)$$

$$\tilde{S}(\gamma)\tilde{S}(I_2) = -\tilde{S}(I_2)\tilde{S}(\gamma), \tilde{S}(\gamma)\tilde{S}(I_4) = -\tilde{S}(I_4)\tilde{S}(\gamma), \tilde{S}(\gamma)\tilde{S}(I_1) = -\tilde{S}(I_3)\tilde{S}(\gamma), \quad (38)$$

$$\begin{aligned}\tilde{S}(t)\tilde{S}(I_1) &= \tilde{S}(I_1)\tilde{S}(t), \tilde{S}(t)\tilde{S}(I_2) = -\tilde{S}(I_4)\tilde{S}(t), \tilde{S}(t)\tilde{S}(I_3) = -\tilde{S}(I_2)\tilde{S}(t), \tilde{S}(t)\tilde{S}(I_4) = \tilde{S}(I_3)\tilde{S}(t), \\ &\quad (39)\end{aligned}$$

plus a unitary condition

$$\tilde{S}(\gamma)^\dagger \tilde{S}(\gamma) = \mathbf{1}, \tilde{S}(t)^\dagger \tilde{S}(t) = \mathbf{1} \quad (40)$$

Note that we add a minus sign to the second and the third equations in Eq.(39) compared with Eq.(35) according to the same physics convention, though they are completely equivalent.

Solving Eqs.(37)...(40) for four-dimensional case gives two solutions

$$\underline{4}_+ : \tilde{S}(\gamma) = \Gamma \cdot \tilde{\gamma}_+, \Gamma = \frac{1}{\sqrt{2}} \cdot \begin{pmatrix} 0 & 0 & 1 & -1 \\ 0 & 0 & -1 & -1 \\ 1 & -1 & 0 & 0 \\ -1 & -1 & 0 & 0 \end{pmatrix}, \tilde{\gamma}_+ = i$$

$$\tilde{S}(t) = T \cdot \tilde{t}, T = \frac{1}{\sqrt{2}} \cdot \begin{pmatrix} 1 & -i & 0 & 0 \\ 1 & i & 0 & 0 \\ 0 & 0 & i & 1 \\ 0 & 0 & -i & 1 \end{pmatrix}, \tilde{t} = e^{i\frac{7\pi}{4}}$$

$$\underline{4}_- : \tilde{S}(\gamma) = \Gamma \cdot \tilde{\gamma}_-, \tilde{\gamma}_- = -i, \tilde{S}(t) = T \cdot \tilde{t}$$

Note that $\underline{4}_+$ is just the representation S with $\tilde{S}(I_2) = -S(I_2)$.

As for eight-dimensional case we can suppose

$$\tilde{\gamma} = \begin{pmatrix} 0 & \tilde{c} \\ \tilde{d} & 0 \end{pmatrix}, \tilde{t} = \begin{pmatrix} \tilde{a} & 0 \\ 0 & \tilde{d} \end{pmatrix}$$

The solution $\underline{8}$ is given by

$$\tilde{c} = e^{i\frac{\pi}{3}}, \tilde{d} = e^{i\frac{2\pi}{3}}, \tilde{a} = e^{i\frac{5\pi}{12}}, \tilde{d} = e^{i\frac{13\pi}{12}}$$

Finally, we set for the twelve-dimensional case

$$\tilde{\gamma} = \begin{pmatrix} 0 & 0 & \tilde{z} \\ 0 & \tilde{y} & 0 \\ \tilde{x} & 0 & 0 \end{pmatrix}, \tilde{t} = \begin{pmatrix} 0 & 0 & \tilde{n} \\ \tilde{l} & 0 & 0 \\ 0 & \tilde{m} & 0 \end{pmatrix}$$

Such that

$$\underline{12}_+ : \tilde{x} = 1, \tilde{y} = i, \tilde{z} = -1, \tilde{l} = 1, \tilde{m} = 1, \tilde{n} = e^{i\frac{5\pi}{4}}$$

$$\underline{12}_- : \tilde{x} = 1, \tilde{y} = -i, \tilde{z} = -1, \tilde{l} = 1, \tilde{m} = -1, \tilde{n} = e^{i\frac{\pi}{4}}$$

So far, we obtain all inequivalent irreducible representations of $\overline{O_4}$ and we summarize our results in Table IV.

V Structure of $\overline{SO_4}$

Specify $n = 4$ in Eq.(21) then we know immediately that $|SO_4| = 192$. Introduce

$$\eta = \gamma t \gamma t^2 \gamma; \quad (41)$$

$$\alpha = (t^2 \gamma)^2, \beta = t \gamma t^2 \gamma t; \quad (42)$$

$$x = e_1 \eta, y = e_4 \eta, q = e_2 \eta \quad (43)$$

then the structure of SO_4 can be summarized as

$$x^2 = y^2 = q^4 = e, yx = xy, qx = xq^3, qy = yq^3 \quad (44)$$

$$\alpha^2 = \beta^2 = t^3 = e, \beta\alpha = \alpha\beta, t\alpha = \alpha\beta t, t\beta = \alpha t \quad (45)$$

$$\alpha x = q\beta, \alpha y = q^3\beta, \alpha q = x\beta \quad (46)$$

$$\beta x = q^3\alpha, \beta y = q\alpha, \beta q = y\alpha \quad (47)$$

$$tx = xt^2, ty = q^3 t^2, tq = yt^2 \quad (48)$$

together with

$$x\alpha = \beta q^3, xt = t^2 x; y\beta = \alpha q^3, yt = t^2 y; qt = t^2 y \quad (49)$$

Accordingly, each group element can be expressed as a “normal ordering” product of $x, y \rightarrow q \rightarrow \alpha, \beta \rightarrow t$ and their powers from left to right. Throwing away all classes which belong to O_4 but not to SO_4 , there are 11 left which are 1, 3, 5, 7, 8, 11, 12, 14, 15, 18, 20 in Table I. The 14th and the 20th will part into two classes with equal numbers of elements each under adjoint action of SO_4 which are denoted as 14, 14', 20, 20'. Therefore, there are 13 conjugate classes in SO_4 .

Due to the fact that $\overline{SO}_4 \triangleleft \overline{O}_4$, the diagram

$$\begin{array}{ccccccc} 0 & \longrightarrow & Z_2 & \longrightarrow & \overline{O}_4 & \xrightarrow{\pi} & O_4 \longrightarrow 1 \\ & & \parallel & & \uparrow & & \uparrow \\ 0 & \longrightarrow & Z_2 & \longrightarrow & \overline{SO}_4 & \longrightarrow & SO_4 \longrightarrow 1 \end{array}$$

is commutative. We can lift generating relations (44)(45)...(49) to

$$\begin{aligned} x^2 = y^2 = q^4 = -1, & \quad yx = -xy, qx = -xq^3, qy = -yq^3 \\ \alpha^2 = \beta^2 = t^3 = -1, & \quad \beta\alpha = -\alpha\beta, t\alpha = -\alpha\beta t, t\beta = \alpha t \\ & \quad \alpha x = q\beta, \alpha y = q^3\beta, \alpha q = x\beta \\ & \quad \beta x = q^3\alpha, \beta y = q\alpha, \beta q = y\alpha \\ & \quad tx = -xt^2, ty = q^3t^2, tq = yt^2 \\ & \quad x\alpha = \beta q^3, xt = -t^2x; y\beta = \alpha q^3, yt = t^2q^3; qt = t^2y \end{aligned}$$

by the definition (41)(42)(43). As to the conjugate classification, 1, 5, 8, 11, 14, 15, 18, 20, 14', 20' violate relation $-g \sim g$, so that \overline{SO}_4 is partitioned into 23 classes, suggesting that there are altogether 23 inequivalent irreducible representations in which 13 representations are single-valued to SO_4 . We summarize the conjugate classes of \overline{SO}_4 in Table V.

VI Representation theory of \overline{SO}_4

All inequivalent irreducible representations of \overline{SO}_4 can be reduced out from those of \overline{O}_4 . Table VI gives the characters of all inequivalent irreducible representations of \overline{O}_4 , respect to the classes of \overline{SO}_4 . A brief observation gives some important information. Firstly, $1_1 \cong 1_4, 1_2 \cong 1_3, 2_1 \cong 2_2, 3_1 \cong 3_4, 3_2 \cong 3_3, 4_1 \cong 4_4, 4_2 \cong 4_3, 8_1 \cong 8_2, 6_1 \cong 6_3$. Secondly, omitting equivalence, $1_1, 1_2, 2_1, 3_1, 3_1, 4_1, 4_2, 8_1, 6_1$, remain irreducible within \overline{SO}_4 , while other 7 become reducible. Thirdly, as for each of these reducible ones, the inner product of itself's character equals to 2, implying that it can be reduced to two inequivalent irreducible representations, thus there are 13 single-valued and 10 spinor representations as we expected. Finally, it is one possible solution to Burside theorem that each of

these seven reducible representations splits into 2 inequivalent irreducible representations with equal dimensions. We conjecture that it is the solution to our representation theory of \overline{SO}_4 and try to verify it below.

Summarily speaking, there are nine single-valued inequivalent irreducible representations inherited from \overline{O}_4

$$1_1, 1_2, 2 \equiv 2_1, 3_1, 3_2, 4_1, 4_2, 8 \equiv 8_1, 6 \equiv 6_1$$

and we conjecture the splitting relations

$$\begin{aligned} 6_2, 6_4 &\rightarrow 3_\alpha, 3_\beta, 3_\gamma, 3_\delta \\ 4_1, 4_2 &\rightarrow 2_\alpha, 2_\beta, 2_\gamma, 2_\delta \\ \underline{8} &\rightarrow 4_\alpha, 4_\beta \\ 12_1, 12_2 &\rightarrow 6_\alpha, 6_\beta, 6_\gamma, 6_\delta \end{aligned}$$

VI.1 Hidden single-valued representations

The representation matrices of x, y, q in 6_2 are written as

$$x \mapsto \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 & -1 & 0 \end{pmatrix}, y \mapsto \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 & 0 \end{pmatrix}, q \mapsto \begin{pmatrix} -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & -1 & 0 \end{pmatrix}$$

The textures of these matrices inspire us to such a hypotheses that in $3_{\alpha,\beta,\gamma,\delta}$, x, y, q take on a form like

$$x, y \mapsto \begin{pmatrix} \pm 1 & \\ & \pm H \end{pmatrix}, q \mapsto \begin{pmatrix} \pm 1 & \\ & \pm Q \end{pmatrix}$$

where $H \equiv \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$, $Q \equiv \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$. After taking account of the conjugate equivalence, only four possibilities survive from the totally 64, i.e.

$$I : x \rightarrow \begin{pmatrix} 1 & \\ & H \end{pmatrix}, y \rightarrow \begin{pmatrix} 1 & \\ & -H \end{pmatrix}, q \rightarrow \begin{pmatrix} -1 & \\ & Q \end{pmatrix} \quad (50)$$

$$II : x \rightarrow \begin{pmatrix} -1 & \\ & H \end{pmatrix}, y \rightarrow \begin{pmatrix} -1 & \\ & -H \end{pmatrix}, q \rightarrow \begin{pmatrix} 1 & \\ & Q \end{pmatrix} \quad (51)$$

$$III : x \rightarrow \begin{pmatrix} 1 & \\ & -H \end{pmatrix}, y \rightarrow \begin{pmatrix} 1 & \\ & H \end{pmatrix}, q \rightarrow \begin{pmatrix} -1 & \\ & Q \end{pmatrix} \quad (52)$$

$$IV : x \rightarrow \begin{pmatrix} -1 & \\ & -H \end{pmatrix}, y \rightarrow \begin{pmatrix} -1 & \\ & H \end{pmatrix}, q \rightarrow \begin{pmatrix} 1 & \\ & Q \end{pmatrix} \quad (53)$$

which also satisfy Eq.(44). Then we regard α, β, t as unknowns, (45)..(49) as constraint, and solve these matrix equations. Modulo similarity, each of Eqs.(50)(51) (52)(53) gives two solutions, labeled as $I, I', II, II', III, III', IV, IV'$; however, there is no difficulty to find out that $I \cong III$, $I' \cong III'$, $II \cong IV$, $II' \cong IV'$. Thus I, I', II, II' are what we need.

$$\begin{aligned} 3_\alpha \equiv I : \alpha &\rightarrow \text{diag}(-1, 1, -1), \beta \rightarrow \text{diag}(1, -1, -1), t \rightarrow \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix} \\ 3_\beta \equiv I' : \alpha &\rightarrow \text{diag}(1, -1, -1), \beta \rightarrow \text{diag}(-1, -1, 1), t \rightarrow \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix} \\ 3_\gamma \equiv II : \alpha &\rightarrow \text{diag}(-1, 1, -1), \beta \rightarrow \text{diag}(1, -1, -1), t \rightarrow \begin{pmatrix} 0 & 0 & 1 \\ -1 & 0 & 0 \\ 0 & -1 & 0 \end{pmatrix} \end{aligned}$$

$$3_\delta \equiv II' : \alpha \rightarrow \text{diag}(1, -1, -1), \beta \rightarrow \text{diag}(-1, -1, 1), t \rightarrow \begin{pmatrix} 0 & 0 & 1 \\ -1 & 0 & 0 \\ 0 & -1 & 0 \end{pmatrix}$$

VI.2 Spinor representations

It is more straightforward to reduce out the spinor representations. We recall that the spinor representation matrices of O_4 are of the form of a tensor product

$$S_i(g) = S(g) \otimes s_i(g), \forall g \in \overline{O_4}, i = \underline{4}_1, \underline{4}_2, \underline{8}, \underline{12}_1, \underline{12}_2$$

where S is given by the algebraic isomorphism from $Cl(E^4)$ to $M_2(\mathbf{H})$ and s_i has the same texture (zero matrix element positions) of irreducible representation i of S_4 . Additionally, for g in $\overline{SO_4}$, $S(g)$ takes on a 2-by-2 block diagonal form

$$S(g) = \begin{pmatrix} S_{up}(g) & 0 \\ 0 & S_{down}(g) \end{pmatrix} \quad (54)$$

So it is just what we want

$$S_{up}(x) = \frac{1}{\sqrt{2}} \cdot \begin{pmatrix} e^{i(\frac{3}{4})\pi} & e^{i(\frac{-3}{4})\pi} \\ e^{i(\frac{1}{4})\pi} & e^{i(\frac{-1}{4})\pi} \end{pmatrix}, S_{up}(y) = \frac{1}{\sqrt{2}} \cdot \begin{pmatrix} e^{i(\frac{3}{4})\pi} & e^{i(\frac{1}{4})\pi} \\ e^{i(\frac{-3}{4})\pi} & e^{i(\frac{-1}{4})\pi} \end{pmatrix} \quad (55)$$

$$S_{up}(q) = \frac{1}{\sqrt{2}} \cdot \begin{pmatrix} e^{i(\frac{1}{4})\pi} & e^{i(\frac{3}{4})\pi} \\ e^{i(\frac{3}{4})\pi} & e^{i(\frac{1}{4})\pi} \end{pmatrix}, S_{up}(t) = \frac{1}{\sqrt{2}} \cdot \begin{pmatrix} 1 & -i \\ 1 & i \end{pmatrix} \quad (56)$$

$$S_{up}(\alpha) = \begin{pmatrix} 0 & -i \\ -i & 0 \end{pmatrix}, S_{up}(\beta) = \begin{pmatrix} -i & 0 \\ 0 & i \end{pmatrix} \quad (57)$$

$$S_{down}(x) = \frac{1}{\sqrt{2}} \cdot \begin{pmatrix} e^{i(\frac{3}{4})\pi} & e^{i(\frac{-3}{4})\pi} \\ e^{i(\frac{1}{4})\pi} & e^{i(\frac{-1}{4})\pi} \end{pmatrix}, S_{down}(y) \rightarrow \frac{1}{\sqrt{2}} \cdot \begin{pmatrix} e^{i(\frac{-1}{4})\pi} & e^{i(\frac{-3}{4})\pi} \\ e^{i(\frac{1}{4})\pi} & e^{i(\frac{3}{4})\pi} \end{pmatrix} \quad (58)$$

$$S_{down}(q) = \frac{1}{\sqrt{2}} \cdot \begin{pmatrix} e^{i(\frac{1}{4})\pi} & e^{i(\frac{3}{4})\pi} \\ e^{i(\frac{3}{4})\pi} & e^{i(\frac{1}{4})\pi} \end{pmatrix}, S_{down}(t) = \frac{1}{\sqrt{2}} \cdot \begin{pmatrix} i & 1 \\ -i & 1 \end{pmatrix} \quad (59)$$

$$S_{down}(\alpha) = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, S_{down}(\beta) = \begin{pmatrix} 0 & -i \\ -i & 0 \end{pmatrix} \quad (60)$$

Keeping the second factor unchanged, each spinor representation in $\overline{O_4}$ splits into two spinor representations in $\overline{SO_4}$, denoted as $2_\alpha, 2_\beta, 2_\gamma, 2_\delta, 4_\alpha, 4_\beta, 6_\alpha, 6_\beta, 6_\gamma, 6_\delta$.

In fact, $S(g)$ falls in the so-called “chiral”-representation of $Cl(E^4)$ in physical language. Due to $Cl(V) = Cl(V)_e \oplus Cl(V)_o$, and the choice of chiral-representation, there are

$$Cl(E^4)_e \cong \begin{pmatrix} \mathbf{H} & O \\ 0 & \mathbf{H} \end{pmatrix}, Cl(E^4)_o \cong \begin{pmatrix} 0 & \mathbf{H} \\ \mathbf{H} & 0 \end{pmatrix}$$

Notice that $\overline{SO_4} < Spin(4) \subset Cl(E^4)_e$, so our reducing process for spinor representations roots in the structure of Clifford algebra.

Conclusively, our conjecture gives all inequivalent irreducible representations of $\overline{SO_4}$ whose characters are summarized in Table VII.

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A Fundamental lemma of n-dimensional Euclidean geometry

Lemma A-1 (Weak form) *Let $p_i, i = 0, 1, 2, \dots, n$ be n points in n -dimensional Euclidean space E^n which are non-collinear and given n non-negative real numbers $d_i, i = 0, 1, 2, \dots, n$, then there exists at most one point $p \in E^n$ s.t. $d(p, p_i) = d_i$.*

Proof:

Without losing generality, set $p_0 = (0, 0, \dots, 0)$ and understand $p, p_i, i = 1, 2, \dots, n$ as vectors in E^n . Consider equation set

$$(p - p_i, p - p_i) = d_i^2, i = 1, 2, \dots, n \quad (\text{A-1})$$

$$(p, p) = d_0^2 \quad (\text{A-2})$$

Substitute (A-2) into (A-1)

$$(p_i, p) = \frac{1}{2}(d_0^2 - d_i^2 + (p_i, p_i)), i = 1, 2, \dots, n \quad (\text{A-3})$$

The non-collinearity implies that (A-3) has a solution p . The weak form of *fundamental lemma of Euclidean geometry* follows.

□

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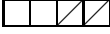
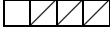
















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Table List

- I. Conjugate Classes of $Z_2^4 \triangleright \triangleleft S_4$
- II. Conjugate Classes of \overline{O}_4
- III. Character Table of Z_2^4
- IV. Character Table of \overline{O}_4
- V. Conjugate Classes of \overline{SO}_4
- VI. Character Table of \overline{O}_4 (with respect to the classes of \overline{SO}_4)
- VII. Character Table of \overline{SO}_4

Table I: Conjugate Classes of $Z_2^4 \rtimes S_4$

No	SplitNo	YoungDiagram	ord	num	det	No	SplitNo	YoungDiagram	ord	num	det
1	1-1		1	1	1	2	1-2		2	4	-1
3	1-3		2	6	1	4	1-4		2	4	-1
5	1-5		2	1	1						
6	2-1		2	12	-1	7	2-2		2	24	1
8	2-3		4	12	1	9	2-4		2	12	-1
10	2-5		4	24	-1	11	2-6		4	12	1
12	3-1		2	12	1	13	3-2		4	24	-1
14	3-3		4	12	1						
15	4-1		3	32	1	16	4-2		6	32	-1
17	4-3		6	32	-1	18	4-4		6	32	1
19	5-1		4	48	-1	20	5-2		8	48	1

“SplitNo” reflects the relation between the classes of $Z_2 \wr S_4$ and those of S_4 . “ord” means order of each class. “num” is the number of elements in each class. “det” is the signature of each class. See Eqs.(16)...(19).

Table II: Conjugate Classes of \overline{O}_4

No.	1	1'	2	3	4	5	6	7	8	8'	9	10	11	12	13	14	14'	15	15'	16	17	18	19	20	20'
num.	1	1	8	12	8	2	24	48	12	12	24	48	24	24	48	12	12	32	32	64	64	64	96	48	48
ord.	1	2	4	4	2	2	4	4	8	8	2	8	8	4	8	4	4	6	3	12	6	6	4	8	8

The labels of classes are descended from those of O_4 with “ \prime ” for those classes split when lifting into \overline{O}_4 .

Table III: Character Table of Z_2^4

\overline{Z}_2^4	$[e]$	$[I_1]$	$[I_2]$	$[I_3]$	$[I_4]$	$[I_{12}]$	$[I_{13}]$	$[I_{14}]$	$[I_{23}]$	$[I_{24}]$	$[I_{34}]$	$[I_{234}]$	$[I_{134}]$	$[I_{124}]$	$[I_{123}]$	$[I_{1234}]$
χ_{0000}	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1
χ_{0001}	1	1	1	1	-1	1	1	-1	1	-1	-1	-1	-1	-1	1	-1
χ_{0010}	1	1	1	-1	1	1	-1	1	-1	1	-1	-1	-1	1	-1	-1
χ_{0100}	1	1	-1	1	1	-1	1	1	-1	-1	1	-1	1	-1	-1	-1
χ_{1000}	1	-1	1	1	1	-1	-1	-1	1	1	1	1	-1	-1	-1	-1
χ_{0011}	1	1	1	-1	-1	1	-1	-1	-1	-1	1	1	1	-1	-1	1
χ_{0101}	1	1	-1	1	-1	-1	1	-1	-1	1	-1	1	-1	1	-1	1
χ_{1001}	1	-1	1	1	-1	-1	-1	1	1	-1	-1	-1	1	1	-1	1
χ_{0110}	1	1	-1	-1	1	-1	-1	1	1	-1	-1	1	-1	-1	1	1
χ_{1010}	1	-1	1	-1	1	-1	1	-1	-1	1	-1	-1	1	-1	1	1
χ_{1100}	1	-1	-1	1	1	1	-1	-1	-1	-1	1	-1	-1	1	1	1
χ_{1110}	1	-1	-1	-1	1	1	1	-1	1	-1	-1	1	1	1	-1	-1
χ_{1101}	1	-1	-1	1	-1	1	-1	1	-1	1	-1	1	1	-1	1	-1
χ_{1011}	1	-1	1	-1	-1	-1	1	1	-1	-1	1	1	-1	1	1	-1
χ_{0111}	1	1	-1	-1	-1	-1	-1	-1	1	1	1	-1	1	1	1	-1
χ_{1111}	1	-1	-1	-1	-1	1	1	1	1	1	1	-1	-1	-1	-1	1

$I_{i_1 i_2 \dots i_a} := I_{i_1} \cdot I_{i_2} \cdot \dots \cdot I_{i_a}$. Irreducible characters are labeled as $\chi_{s_1 s_2 s_3 s_4}$, $s_i \in \mathbb{Z}/2\mathbb{Z}$ (see Eq.(20)).

Table IV: Character Table of \overline{O}_4

	Π_S					Π_P					Π_V			Π_A			Π_T				spinor rep.				
	[4]	[1 ⁴]	[2 ²]	[31]	[21 ²]	[4]	[1 ⁴]	[2 ²]	[31]	[21 ²]	[3]	[1 ³]	[21]	[3]	[1 ³]	[21]	π_{00}	π_{01}	π_{10}	π_{11}	$\underline{4}_+$	$\underline{4}_-$	$\underline{8}$	$\underline{12}_+$	$\underline{12}_-$
	$\chi_1^{(1)}$	$\chi_3^{(1)}$	$\chi_1^{(2)}$	$\chi_1^{(3)}$	$\chi_3^{(3)}$	$\chi_2^{(1)}$	$\chi_4^{(1)}$	$\chi_2^{(2)}$	$\chi_2^{(3)}$	$\chi_4^{(3)}$	$\chi_1^{(4)}$	$\chi_3^{(4)}$	$\chi_1^{(8)}$	$\chi_4^{(4)}$	$\chi_2^{(4)}$	$\chi_2^{(8)}$	$\chi_3^{(6)}$	$\chi_2^{(6)}$	$\chi_4^{(6)}$	$\chi_1^{(6)}$	$\chi_1^{(4)}$	$\chi_2^{(4)}$	$\chi^{(8)}$	$\chi_1^{(12)}$	$\chi_2^{(12)}$
1	1	1	2	3	3	1	1	2	3	3	4	4	8	4	4	8	6	6	6	6	4	4	8	12	12
1'	1	1	2	3	3	1	1	2	3	3	4	4	8	4	4	8	6	6	6	6	-4	-4	-8	-12	-12
2	1	1	2	3	3	-1	-1	-2	-3	-3	2	2	4	-2	-2	-4	0	0	0	0	0	0	0	0	0
3	1	1	2	3	3	1	1	2	3	3	0	0	0	0	0	0	-2	-2	-2	-2	0	0	0	0	0
4	1	1	2	3	3	-1	-1	-2	-3	-3	-2	-2	-4	2	2	4	0	0	0	0	0	0	0	0	0
5	1	1	2	3	3	1	1	2	3	3	-4	-4	-8	-4	-4	-8	6	6	6	6	0	0	0	0	0
6	1	-1	0	1	-1	1	-1	0	1	-1	2	-2	0	2	-2	0	2	0	-2	0	0	0	0	0	0
7	1	-1	0	1	-1	-1	1	0	-1	1	0	0	0	0	0	0	0	2	0	-2	0	0	0	0	0
8	1	-1	0	1	-1	-1	1	0	-1	1	2	-2	0	-2	2	0	0	-2	0	2	$-2\sqrt{2}$	$2\sqrt{2}$	0	$-2\sqrt{2}$	$2\sqrt{2}$
8'	1	-1	0	1	-1	-1	1	0	-1	1	2	-2	0	-2	2	0	0	-2	0	2	$2\sqrt{2}$	$-2\sqrt{2}$	0	$2\sqrt{2}$	$-2\sqrt{2}$
9	1	-1	0	1	-1	1	-1	0	1	-1	-2	2	0	-2	2	0	2	0	-2	0	0	0	0	0	0
10	1	-1	0	1	-1	1	-1	0	1	-1	0	0	0	0	0	0	-2	0	2	0	0	0	0	0	0
11	1	-1	0	1	-1	-1	1	0	-1	1	-2	2	0	2	-2	0	0	-2	0	2	0	0	0	0	0
12	1	1	2	-1	-1	1	1	2	-1	-1	0	0	0	0	0	0	2	-2	2	-2	0	0	0	0	0
13	1	1	2	-1	-1	-1	-1	-2	1	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
14	1	1	2	-1	-1	1	1	2	-1	-1	0	0	0	0	0	0	-2	2	-2	2	2	2	4	-2	-2
14'	1	1	2	-1	-1	1	1	2	-1	-1	0	0	0	0	0	0	-2	2	-2	2	-2	-2	-4	2	2
15	1	1	-1	0	0	1	1	-1	0	0	1	1	-1	1	1	-1	0	0	0	0	2	2	-2	0	0
15'	1	1	-1	0	0	1	1	-1	0	0	1	1	-1	1	1	-1	0	0	0	0	-2	-2	2	0	0
16	1	1	-1	0	0	-1	-1	1	0	0	-1	-1	1	1	1	-1	0	0	0	0	0	0	0	0	0
17	1	1	-1	0	0	-1	-1	1	0	0	1	1	-1	-1	-1	1	0	0	0	0	0	0	0	0	0
18	1	1	-1	0	0	1	1	-1	0	0	-1	-1	1	-1	-1	1	0	0	0	0	0	0	0	0	0
19	1	-1	0	-1	1	1	-1	0	-1	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
20	1	-1	0	-1	1	-1	1	0	1	-1	0	0	0	0	0	0	0	0	0	0	$\sqrt{2}$	$-\sqrt{2}$	0	$-\sqrt{2}$	$\sqrt{2}$
20'	1	-1	0	-1	1	-1	1	0	1	-1	0	0	0	0	0	0	0	0	0	0	$-\sqrt{2}$	$\sqrt{2}$	0	$\sqrt{2}$	$-\sqrt{2}$

Character is labeled by a superscript showing its dimension where an underline shows a spinor representation and a subscript distinguishing different representations with same dimension.

Table V: Conjugate Classes of \overline{SO}_4

No.	1	$\bar{1}$	3	5	$\bar{5}$	7	8	$\bar{8}$	11	$\bar{11}$	12	14	$\bar{14}$	14'	$\bar{14}'$	15	$\bar{15}$	18	$\bar{18}$	20	$\bar{20}$	20'	$\bar{20}'$
num.	1	1	12	1	1	48	12	12	12	12	24	6	6	6	6	32	32	32	32	24	24	24	24
ord.	1	2	4	2	2	4	8	8	8	8	4	4	4	4	4	6	3	6	6	8	8	8	8

Table VI: Character Table of $\overline{O_4}$ (with respect to the classes of $\overline{SO_4}$)

	1	$\bar{1}$	3	5	$\bar{5}$	7	8	$\bar{8}$	11	$\bar{11}$	12	14	$\bar{14}$	14'	$\bar{14}'$	15	$\bar{15}$	18	$\bar{18}$	20	$\bar{20}$	20'	$\bar{20}'$	(χ, χ)
$\chi_1^{(1)}$	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1
$\chi_2^{(1)}$	1	1	1	1	1	-1	-1	-1	-1	-1	1	1	1	1	1	1	1	1	1	-1	-1	-1	-1	1
$\chi_1^{(2)}$	2	2	2	2	2	0	0	0	0	0	2	2	2	2	2	-1	-1	-1	-1	0	0	0	0	1
$\chi_1^{(3)}$	3	3	3	3	3	1	1	1	1	1	-1	-1	-1	-1	-1	0	0	0	0	-1	-1	-1	-1	1
$\chi_2^{(3)}$	3	3	3	3	3	-1	-1	-1	-1	-1	-1	-1	-1	-1	-1	0	0	0	0	1	1	1	1	1
$\chi_3^{(1)}$	1	1	1	1	1	-1	-1	-1	-1	-1	1	1	1	1	1	1	1	1	1	-1	-1	-1	-1	1
$\chi_4^{(1)}$	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1
$\chi_2^{(2)}$	2	2	2	2	2	0	0	0	0	0	2	2	2	2	2	-1	-1	-1	-1	0	0	0	0	1
$\chi_3^{(3)}$	3	3	3	3	3	-1	-1	-1	-1	-1	-1	-1	-1	-1	-1	0	0	0	0	1	1	1	1	1
$\chi_4^{(3)}$	3	3	3	3	3	1	1	1	1	1	-1	-1	-1	-1	-1	0	0	0	0	-1	-1	-1	-1	1
$\chi_1^{(4)}$	4	4	0	-4	-4	0	2	2	-2	-2	0	0	0	0	0	1	1	-1	-1	0	0	0	0	1
$\chi_2^{(4)}$	4	4	0	-4	-4	0	-2	-2	2	2	0	0	0	0	0	1	1	-1	-1	0	0	0	0	1
$\chi_1^{(8)}$	8	8	0	-8	-8	0	0	0	0	0	0	0	0	0	0	-1	-1	1	1	0	0	0	0	1
$\chi_3^{(4)}$	4	4	0	-4	-4	0	-2	-2	2	2	0	0	0	0	0	1	1	-1	-1	0	0	0	0	1
$\chi_4^{(4)}$	4	4	0	-4	-4	0	2	2	-2	-2	0	0	0	0	0	1	1	-1	-1	0	0	0	0	1
$\chi_2^{(8)}$	8	8	0	-8	-8	0	0	0	0	0	0	0	0	0	0	-1	-1	1	1	0	0	0	0	1
$\chi_1^{(6)}$	6	6	-2	6	6	0	0	0	0	0	2	-2	-2	-2	-2	0	0	0	0	0	0	0	0	1
$\chi_2^{(6)}$	6	6	-2	6	6	2	-2	-2	-2	-2	-2	2	2	2	2	0	0	0	0	0	0	0	0	2
$\chi_3^{(6)}$	6	6	-2	6	6	0	0	0	0	0	2	-2	-2	-2	-2	0	0	0	0	0	0	0	0	1
$\chi_4^{(6)}$	6	6	-2	6	6	-2	2	2	2	2	-2	2	2	2	2	0	0	0	0	0	0	0	0	2
$\chi_1^{(4)}$	4	-4	0	0	0	0	$-2\sqrt{2}$	$2\sqrt{2}$	0	0	0	2	-2	2	-2	2	-2	0	0	$\sqrt{2}$	$-\sqrt{2}$	$\sqrt{2}$	$-\sqrt{2}$	2
$\chi_2^{(4)}$	4	-4	0	0	0	0	$2\sqrt{2}$	$-2\sqrt{2}$	0	0	0	2	-2	2	-2	2	-2	0	0	$-\sqrt{2}$	$\sqrt{2}$	$-\sqrt{2}$	$\sqrt{2}$	2
$\chi^{(8)}$	8	-8	0	0	0	0	0	0	0	0	0	4	-4	4	-4	-2	2	0	0	0	0	0	0	2
$\chi_1^{(12)}$	12	-12	0	0	0	0	$-2\sqrt{2}$	$2\sqrt{2}$	0	0	0	-2	2	-2	2	0	0	0	0	$-\sqrt{2}$	$\sqrt{2}$	$-\sqrt{2}$	$\sqrt{2}$	2
$\chi_2^{(12)}$	12	-12	0	0	0	0	$2\sqrt{2}$	$-2\sqrt{2}$	0	0	0	-2	2	-2	2	0	0	0	0	$\sqrt{2}$	$-\sqrt{2}$	$\sqrt{2}$	$-\sqrt{2}$	2

(χ, χ) evaluates the inner product of a character with itself.

Table VII: Character Table of \overline{SO}_4

	1	$\bar{1}$	3	5	$\bar{5}$	7	8	$\bar{8}$	11	$\bar{11}$	12	14	$\bar{14}$	$14'$	$\bar{14}'$	15	$\bar{15}$	18	$\bar{18}$	20	$\bar{20}$	$20'$	$\bar{20}'$
num.	1	1	12	1	1	48	12	12	12	12	24	6	6	6	6	32	32	32	32	24	24	24	24
$\chi_1^{(1)}$	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1
$\chi_2^{(1)}$	1	1	1	1	1	-1	-1	-1	-1	-1	1	1	1	1	1	1	1	1	1	-1	-1	-1	-1
$\chi^{(2)}$	2	2	2	2	2	0	0	0	0	0	2	2	2	2	2	-1	-1	-1	-1	0	0	0	0
$\chi_1^{(3)}$	3	3	3	3	3	1	1	1	1	1	-1	-1	-1	-1	-1	0	0	0	0	-1	-1	-1	-1
$\chi_2^{(3)}$	3	3	3	3	3	-1	-1	-1	-1	-1	-1	-1	-1	-1	-1	0	0	0	0	1	1	1	1
$\chi_1^{(4)}$	4	4	0	-4	-4	0	2	2	-2	-2	0	0	0	0	0	1	1	-1	-1	0	0	0	0
$\chi_2^{(4)}$	4	4	0	-4	-4	0	-2	-2	2	2	0	0	0	0	0	1	1	-1	-1	0	0	0	0
$\chi^{(8)}$	8	8	0	-8	-8	0	0	0	0	0	0	0	0	0	0	-1	-1	1	1	0	0	0	0
$\chi^{(6)}$	6	6	-2	6	6	0	0	0	0	0	2	-2	-2	-2	-2	0	0	0	0	0	0	0	0
$\chi_\alpha^{(3)}$	3	3	-1	3	3	1	-1	-1	-1	-1	-1	-1	-1	3	3	0	0	0	0	-1	-1	1	1
$\chi_\beta^{(3)}$	3	3	-1	3	3	1	-1	-1	-1	-1	-1	3	3	-1	-1	0	0	0	0	1	1	-1	-1
$\chi_\gamma^{(3)}$	3	3	-1	3	3	-1	1	1	1	1	-1	-1	-1	3	3	0	0	0	0	1	1	-1	-1
$\chi_\delta^{(3)}$	3	3	-1	3	3	-1	1	1	1	1	-1	3	3	-1	-1	0	0	0	0	-1	-1	1	1
$\chi_\alpha^{(2)}$	2	-2	0	2	-2	0	$-\sqrt{2}$	$\sqrt{2}$	$\sqrt{2}$	$-\sqrt{2}$	0	2	-2	0	0	1	-1	1	-1	0	0	$\sqrt{2}$	$-\sqrt{2}$
$\chi_\beta^{(2)}$	2	-2	0	2	-2	0	$\sqrt{2}$	$-\sqrt{2}$	$-\sqrt{2}$	$\sqrt{2}$	0	2	-2	0	0	1	-1	1	-1	0	0	$-\sqrt{2}$	$\sqrt{2}$
$\chi_\gamma^{(2)}$	2	-2	0	-2	2	0	$-\sqrt{2}$	$\sqrt{2}$	$-\sqrt{2}$	$\sqrt{2}$	0	0	0	-2	2	1	-1	-1	1	$\sqrt{2}$	$-\sqrt{2}$	0	0
$\chi_\delta^{(2)}$	2	-2	0	-2	2	0	$\sqrt{2}$	$-\sqrt{2}$	$\sqrt{2}$	$-\sqrt{2}$	0	0	0	-2	2	1	-1	-1	1	$-\sqrt{2}$	$\sqrt{2}$	0	0
$\chi_\alpha^{(4)}$	4	-4	0	4	-4	0	0	0	0	0	0	4	-4	0	0	-1	1	-1	1	0	0	0	0
$\chi_\beta^{(4)}$	4	-4	0	-4	4	0	0	0	0	0	0	0	0	-4	4	-1	1	1	-1	0	0	0	0
$\chi_\alpha^{(6)}$	6	-6	0	6	-6	0	$-\sqrt{2}$	$\sqrt{2}$	$\sqrt{2}$	$-\sqrt{2}$	0	-2	2	0	0	0	0	0	0	0	0	$-\sqrt{2}$	$\sqrt{2}$
$\chi_\beta^{(6)}$	6	-6	0	6	-6	0	$\sqrt{2}$	$-\sqrt{2}$	$-\sqrt{2}$	$\sqrt{2}$	0	-2	2	0	0	0	0	0	0	0	0	$\sqrt{2}$	$-\sqrt{2}$
$\chi_\gamma^{(6)}$	6	-6	0	-6	6	0	$-\sqrt{2}$	$\sqrt{2}$	$-\sqrt{2}$	$\sqrt{2}$	0	0	0	2	-2	0	0	0	0	$-\sqrt{2}$	$\sqrt{2}$	0	0
$\chi_\delta^{(6)}$	6	-6	0	-6	6	0	$\sqrt{2}$	$-\sqrt{2}$	$\sqrt{2}$	$-\sqrt{2}$	0	0	0	2	-2	0	0	0	0	$\sqrt{2}$	$-\sqrt{2}$	0	0